Commuting operations factorise

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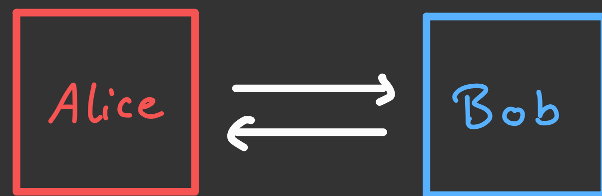
Foundations of Quantum Computing

arXiv:2308.05792
joint work with Renato Renner

When do we have a subsystem structure?

Typical quantum information setting:

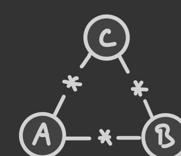


$\mathcal{H}_A$ $\qquad\qquad$ $\mathcal{H}_B$

Examples:

- Quantum cryptography
- Quantum networks
- Quantum computing

Triangle network:



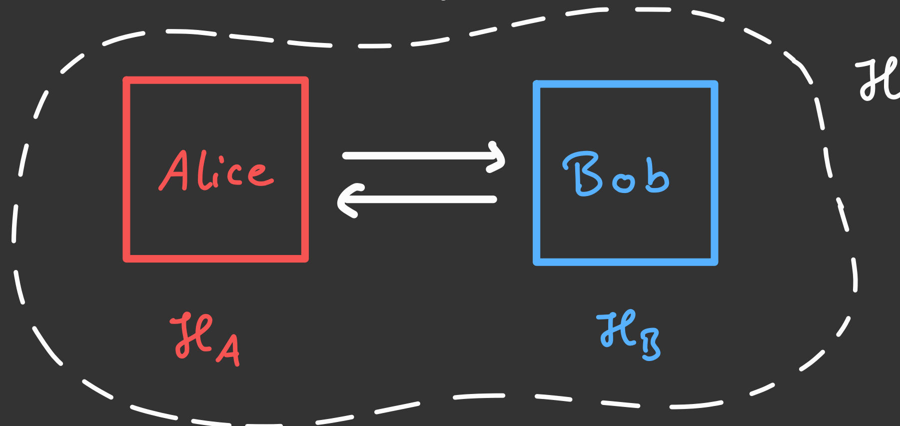
$\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

Quantum computer:

$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$

When do we have a subsystem structure?

Typical quantum information setting:



BUT: The structure of the Hilbert space is not testable

$$\mathcal{H} \overset{?}{=} \mathcal{H}_A \otimes \mathcal{H}_B$$

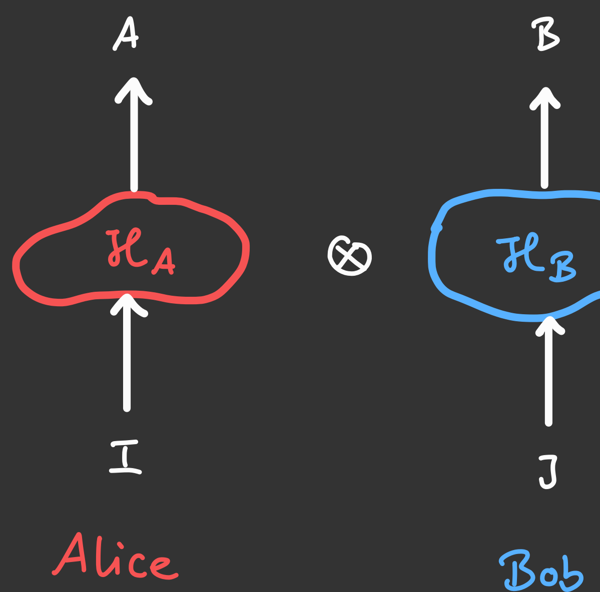
ASSUMPTION

Is this justified?

? ? ? Quantum gravity ? ? ?

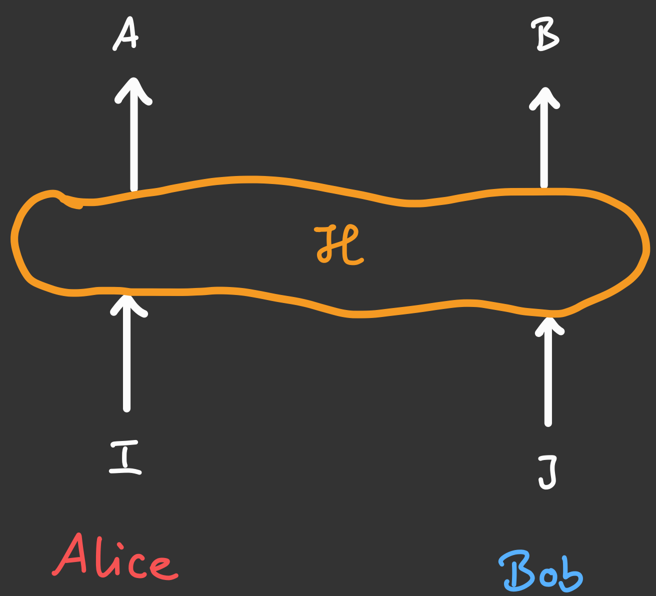
When do we have a subsystem structure?

When do we have this?



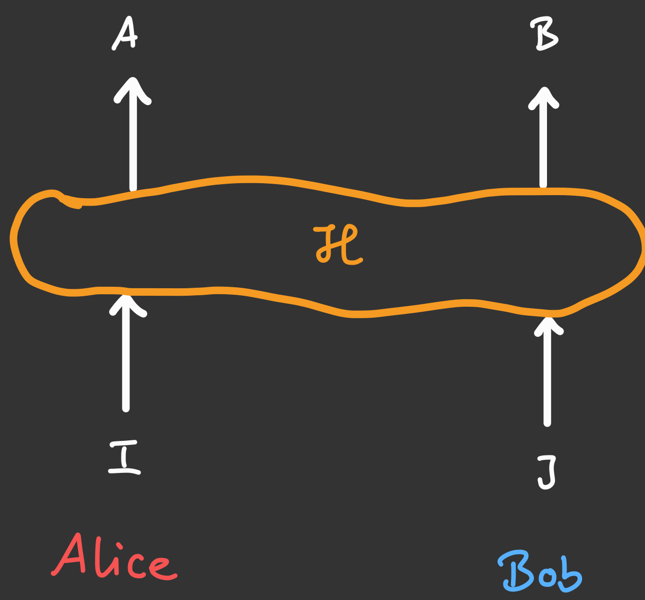
When do we have a subsystem structure?

Starting point:



Make the question more precise

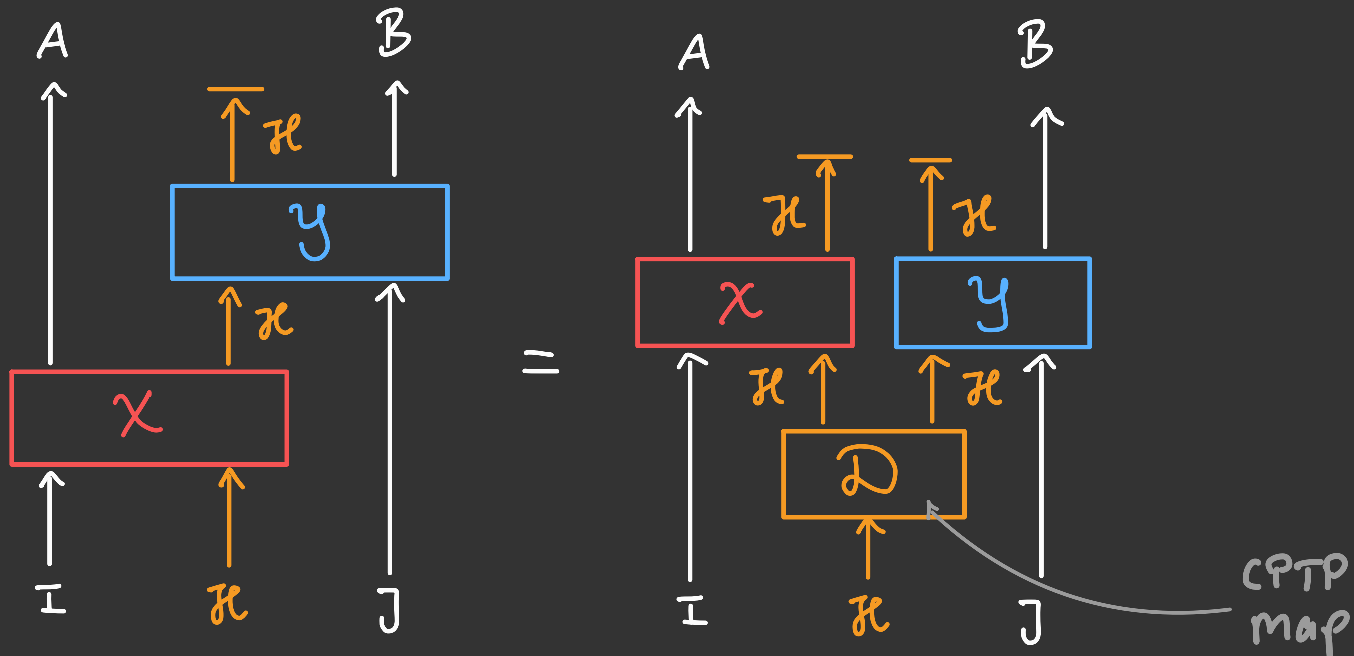
Make the question more precise:



Model Alice and Bob's actions as CPTP maps

Make the question more precise:

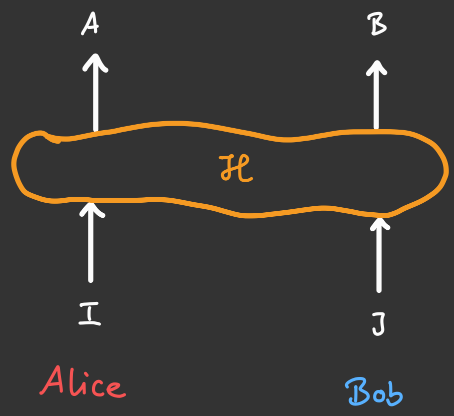
FACTORISATION



Make the question more precise:

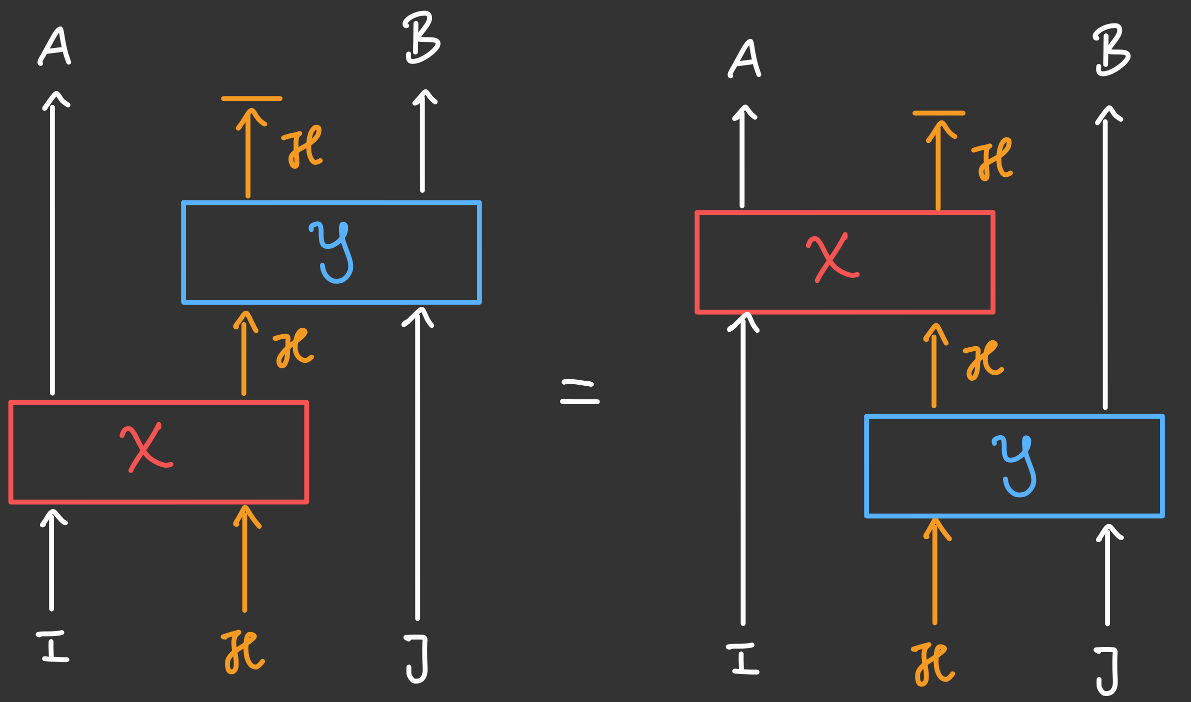
Minimal requirement for factorisation to be possible:

The order in which Alice and Bob perform their actions should not matter.

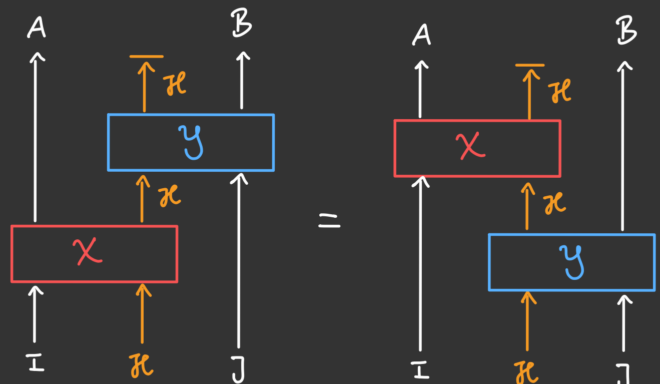


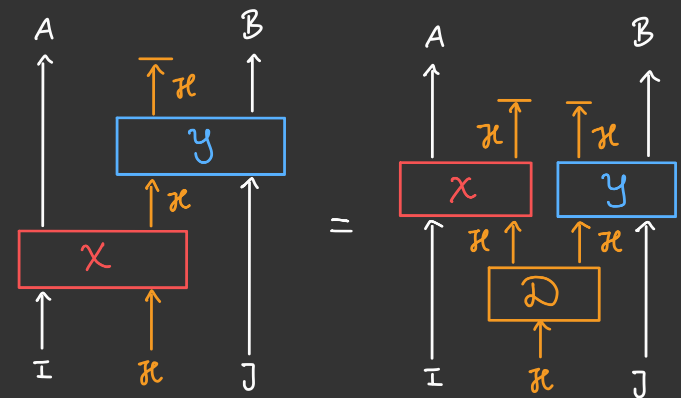
Make the question more precise:

COMMUTATION



Question :



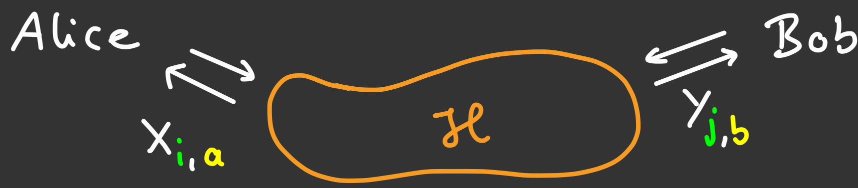


COMMUTATION $\overset{?}{\Longrightarrow}$ FACTORISATION

What we already know: The classical case

What we know: Tsirelson's problem

I, J, A, B classical



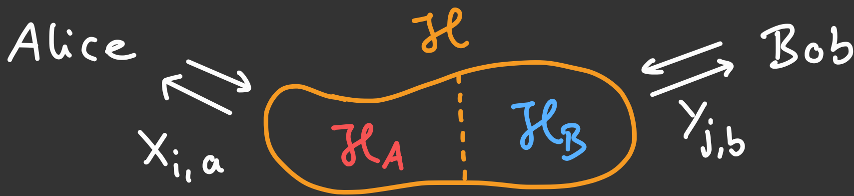
$X_{i,a}$, $Y_{j,b}$ independent projective measurements

For any i : $\{X_{i,a}\}_a$ s.t. $\sum_a X_{i,a} = id_{\mathcal{H}}$

For any j : $\{Y_{i,b}\}_b$ s.t. $\sum_b Y_{j,b} = id_{\mathcal{H}}$

Possible correlations ?

What we know: Tsirelson's problem



I, J, A, B classical

commutation

1) $p(a,b \mid i,j) = \mathrm{tr}(\rho\, X_{i,a} \cdot Y_{j,b})$ with $[X_{i,a}, Y_{j,b}] = 0$

factorisation

2) $p(a,b \mid i,j) = \mathrm{tr}(\rho\, \tilde{X}_{i,a} \otimes \tilde{Y}_{j,b})$ with $\tilde{X}_{i,a} \in \mathcal{B}(\mathcal{H}_A)$

$\tilde{Y}_{j,b} \in \mathcal{B}(\mathcal{H}_B)$

Question: $1) \overset{?}{=} 2)$

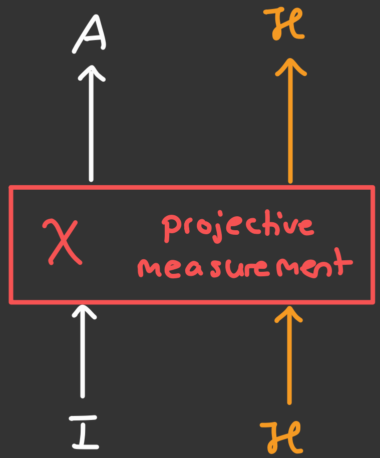
(Tsirelson)

What we know: Tsirelson's problem

I, J, A, B
classical

Map representation:

(Translation to general setting with CPTP maps)



$$\sum_a |a\rangle\langle a|_A \otimes X_{i,a}\, \rho_{\mathcal{H}}\, X_{i,a}$$

$$\Big\uparrow X$$

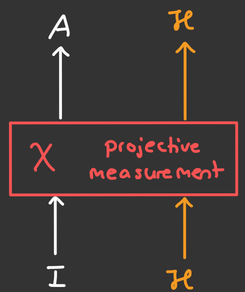
$$|i\rangle\langle i|_I \otimes \rho_{\mathcal{H}}$$

What we know: Tsirelson's problem

I, J, A, B classical

Map representation:

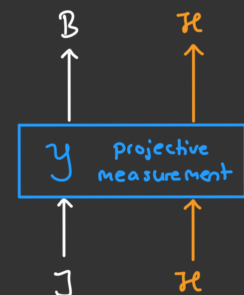
(Translation to general setting with CPTP maps)



$$\sum_a |a\rangle\langle a|_A \otimes X_{i,a}\, \rho_{\mathcal{H}}\, X_{i,a}$$

$$\Big\uparrow\, X$$

$$|i\rangle\langle i|_I \otimes \rho_{\mathcal{H}}$$



$$\sum_b |b\rangle\langle b|_B \otimes Y_{j,b}\, \rho_{\mathcal{H}}\, Y_{j,b}$$

$$\Big\uparrow\, Y$$

$$|j\rangle\langle j|_J \otimes \rho_{\mathcal{H}}$$

What we know: Tsirelson's problem

$\boxed{\text{COMMUTATION}}$

$$\mathcal{X} \circ \mathcal{Y} = \mathcal{Y} \circ \mathcal{X}$$

$$\mathcal{X}\left(|i\rangle\langle i|_I \otimes \rho_{\mathcal{H}}\right) = \sum_a |a\rangle\langle a|_A \otimes X_{i,a}\, \rho_{\mathcal{H}}\, X_{i,a}$$

$$\mathcal{Y}\left(|j\rangle\langle j|_J \otimes \rho_{\mathcal{H}}\right) = \sum_b |b\rangle\langle b|_B \otimes Y_{j,b}\, \rho_{\mathcal{H}}\, Y_{j,b}$$

What we know: Tsirelson's problem

COMMUTATION $\quad [X_{i,a}, Y_{j,b}] = 0 \quad \forall\, i,j,a,b$

FACTORISATION $\quad |i\rangle\langle i| \otimes |j\rangle\langle j| \otimes \rho_{\mathcal{H}}$

$$\mapsto \sum_{a,b} |a\rangle\langle a| \otimes |b\rangle\langle b| \cdot \mathrm{tr}_{\mathcal{H}'\mathcal{H}''}\left(X_{i,a} \otimes Y_{j,b}\, \mathcal{D}(\rho_{\mathcal{H}})\right)$$

$\mathcal{D}: \mathcal{H} \to \mathcal{H}'\mathcal{H}''$ CPTP map $\longrightarrow$ isometry on a larger system $V: \mathcal{H} \to \mathcal{H}'\mathcal{H}''R$

$$\mathrm{tr}\left(X_{i,a} \otimes Y_{j,b}\, \mathcal{D}(\rho_{\mathcal{H}})\right) = \mathrm{tr}\left(X_{i,a} \otimes Y_{j,b} \otimes \mathrm{id}_R\, V \rho_{\mathcal{H}} V^\dagger\right)$$

$$= \mathrm{tr}\left(V^\dagger X_{i,a} \otimes \widetilde{Y}_{j,b}\, V \rho_{\mathcal{H}}\right)$$

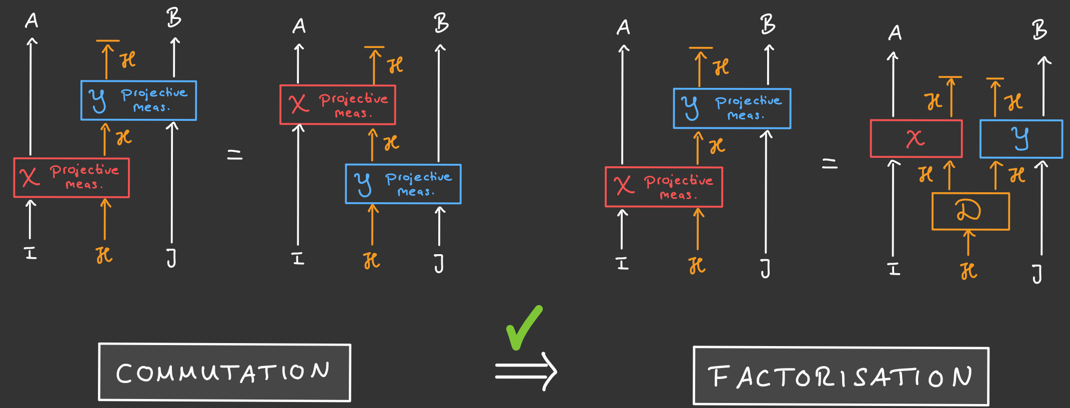
factorisation

$$= \mathrm{tr}\left(X_{i,a} Y_{j,b}\, \rho_{\mathcal{H}}\right)$$

$$\Leftrightarrow \quad X_{i,a} Y_{j,b} = V^\dagger (X_{i,a} \otimes \widetilde{Y}_{j,b}) V$$

$\boxed{\text{Tsirelson's problem}}$

What we know: Tsirelson's problem



I, J, A, B classical

Tsirelson (2006): If $\dim(\mathcal{H}) < \infty$ then $\exists$ isometry V s.t.

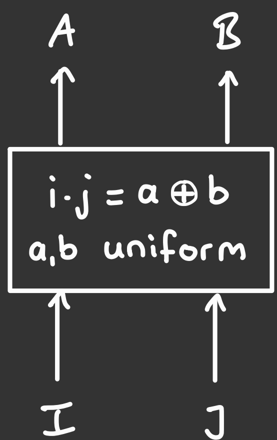
$$X_{i,a} \overset{V}{\cong} \tilde{X}_{i,a} \otimes \mathrm{id} \qquad Y_{j,b} \overset{V}{\cong} \mathrm{id} \otimes \tilde{Y}_{j,b}$$

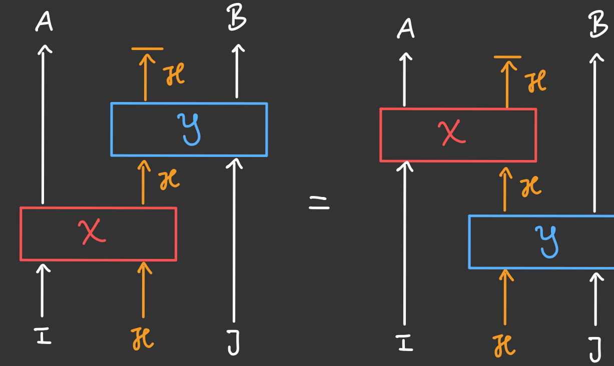
So...

$$\boxed{\text{COMMUTATION}} \overset{?}{\Longrightarrow} \boxed{\text{FACTORISATION}}$$

Not so fast!

PR boxes

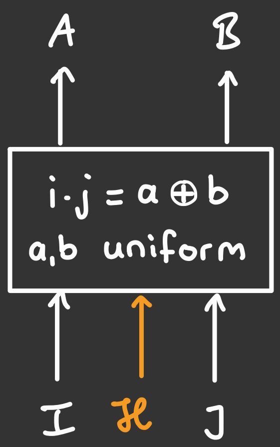




Can we find commuting maps $\mathcal{X}, \mathcal{Y}$ that implement the PR box?

Violate the CHSH inequality with a maximum value of 4

PR boxes

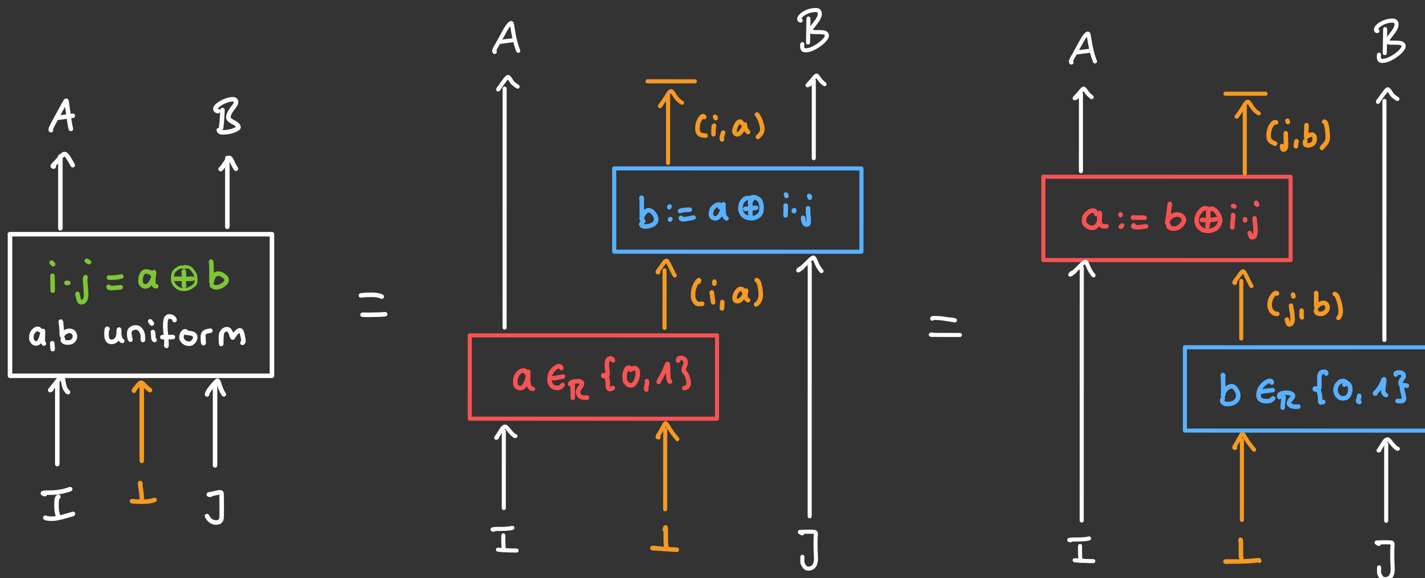


$$\mathcal{H} = \{\perp\} \cup \{0,1\}^2$$

PR boxes

$a \oplus b = a \oplus a \oplus i \cdot j = i \cdot j$ ✓

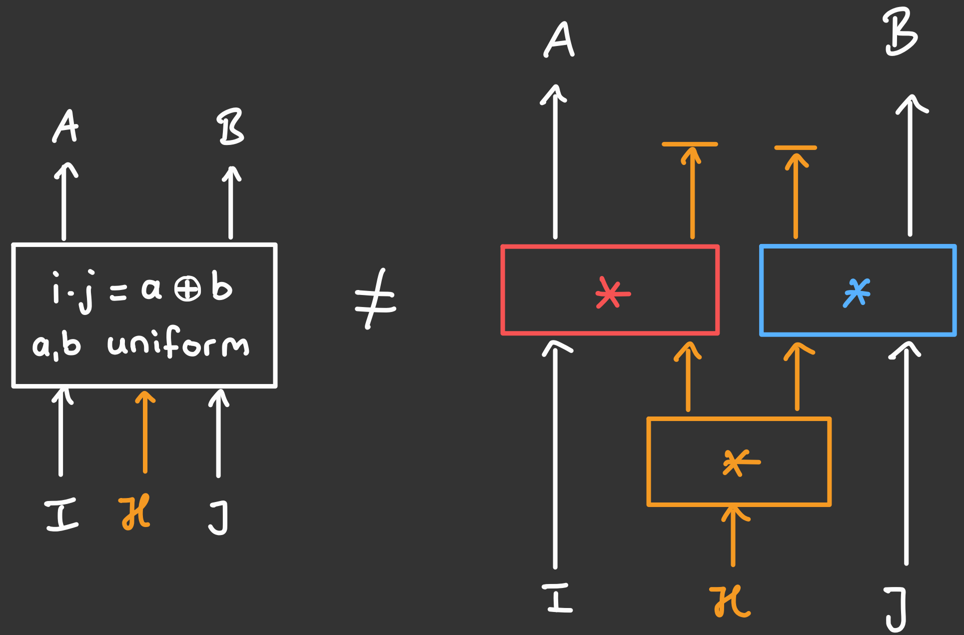
$a \oplus b = b \oplus i \cdot j \oplus b = i \cdot j$ ✓



Output: a,b uniformly random a,b uniformly random

commutation ✓

PR boxes



BUT: PR boxes do not factorise!

(Tsirelson bound 1980)

$CHSH_{PR} = 4$

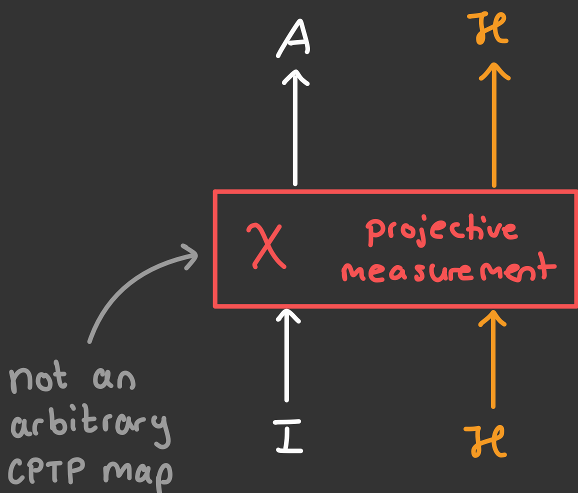
$CHSH_{QM} \leq 2\sqrt{2}$

What does Tsirelson show?

$\boxed{\text{COMMUTATION}} \implies \boxed{\text{FACTORISATION}}$

+ ?

What does Tsirelson show?



$$\sum_a |a\rangle\langle a|_A \otimes X_{i,a}\, \rho_{\mathcal{H}}\, X_{i,a}$$

$$\uparrow \mathcal{X}$$

$$|i\rangle\langle i|_I \otimes \rho_{\mathcal{H}}$$

Update rule: $\rho \longmapsto \rho_a \sim X_{i,a}\, \rho\, X_{i,a}$

Generally: $\rho \longmapsto \rho_a \sim U_{i,a}\, X_{i,a}\, \rho\, X_{i,a}\, U_{i,a}^{\dagger}$

What does Tsirelson show?

$\boxed{\text{COMMUTATION}} \Longrightarrow \boxed{\text{FACTORISATION}}$

+ projective measurements/ POVMs

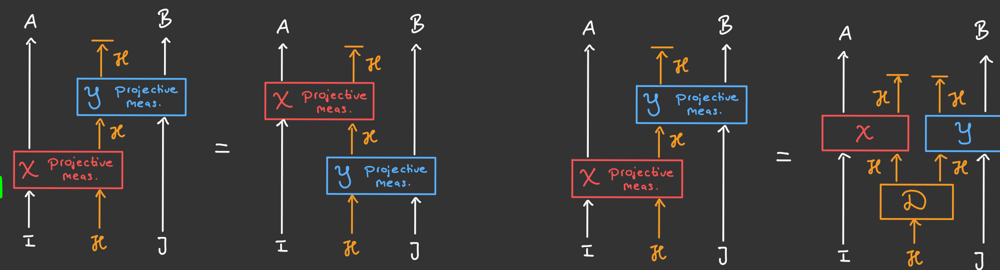
(+ $\dim \mathcal{H} < \infty$)

see [Scholz, Werner 2008]

Fully quantum Tsirelson problem

Fully quantum Tsirelson problem

I, J, A, B
quantum



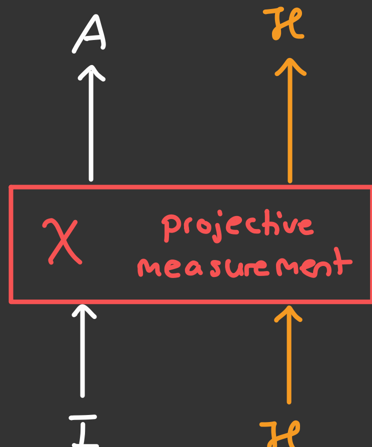
$\boxed{\text{COMMUTATION}}$ $\Longrightarrow$ $\boxed{\text{FACTORISATION}}$

$+ \dim \mathcal{H} < \infty$

"projective measurement"?

Fully quantum Tsirelson problem

"Projective measurement"?



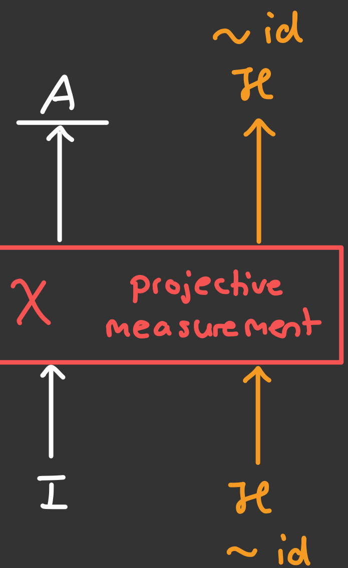
$$\sum_a |a\rangle\langle a|_A \otimes X_{i,a}\, \rho_{\mathcal{H}}\, X_{i,a}$$

$$\Big\uparrow$$

$$|i\rangle\langle i|_I \otimes \rho_{\mathcal{H}}$$

Fully quantum Tsirelson problem

"Projective measurement"?

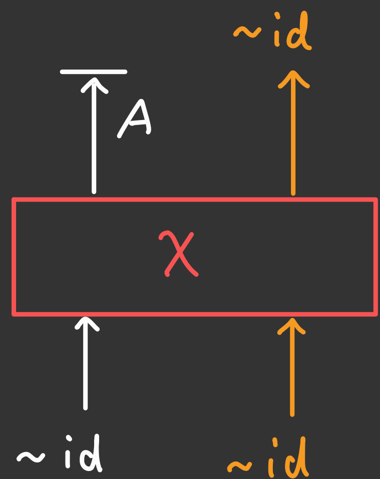


$$|i\rangle\langle i|_I \otimes \underbrace{\rho_{\mathcal{H}}}_{\sim \mathrm{id}} \;\mapsto\; \overbrace{\sum_a X_{i,a}\, \underbrace{\rho_{\mathcal{H}}}_{\sim \mathrm{id}}\, X_{i,a}}^{\sim \mathrm{id}}$$

Fully quantum Tsirelson problem

I, J, A, B
quantum

Unitality condition:



forces the maps to "behave"

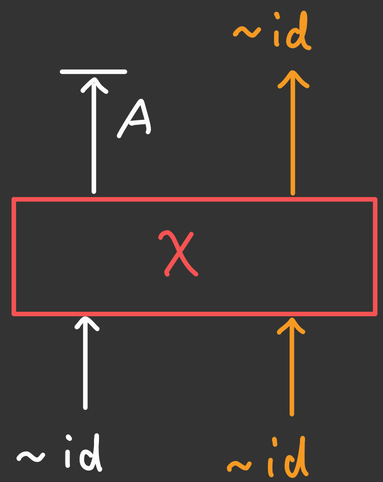
Similar to classical case:

Update rule: $\rho \mapsto \rho_a \sim X_{i,a}\, \rho\, X_{i,a}$

Generally: $\rho \mapsto \rho_a \sim U_{i,a}\, X_{i,a}\, \rho\, X_{i,a}\, U_{i,a}^{\dagger}$

Fully quantum Tsirelson problem

Unitality condition:

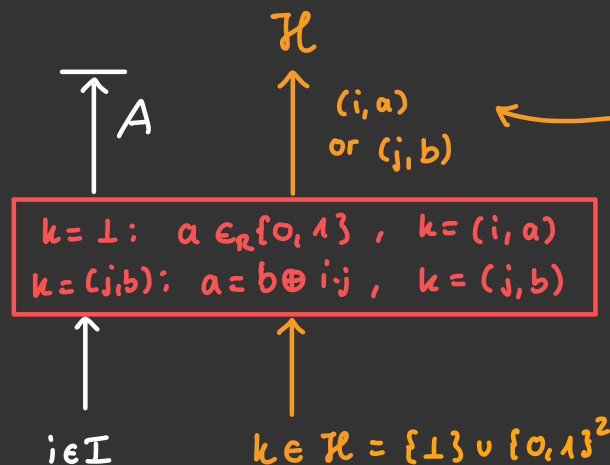


How do PR boxes fail this?

Fully quantum Tsirelson problem

Unitality condition:

PR box:

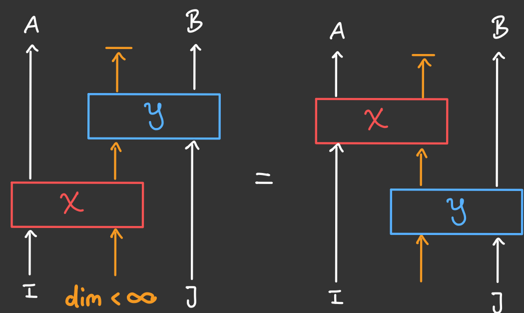


$\Pr[k = \perp] = 0$

<u>cannot</u> be uniform

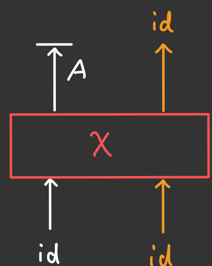
$id_{\mathcal{H}} \hat{=}$ uniform distribution on $\mathcal{H}$

Main result (arXiv: 2308.05792)



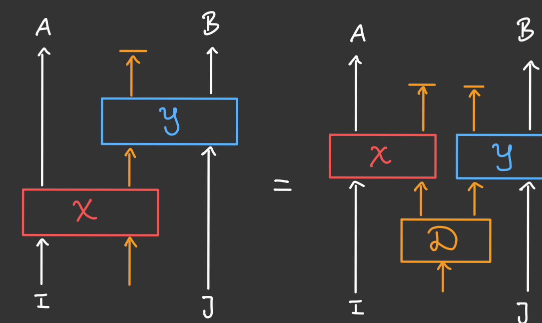
COMMUTATION

and



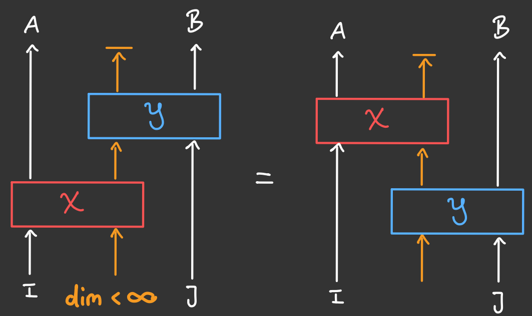
UNITALITY

$\Rightarrow$



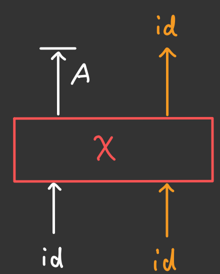
FACTORISATION

Main result (arXiv: 2308.05792)



COMMUTATION

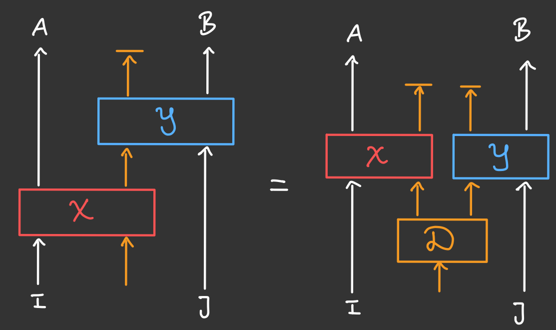
and



UNITALITY

$\Longleftrightarrow$

Converse theorem

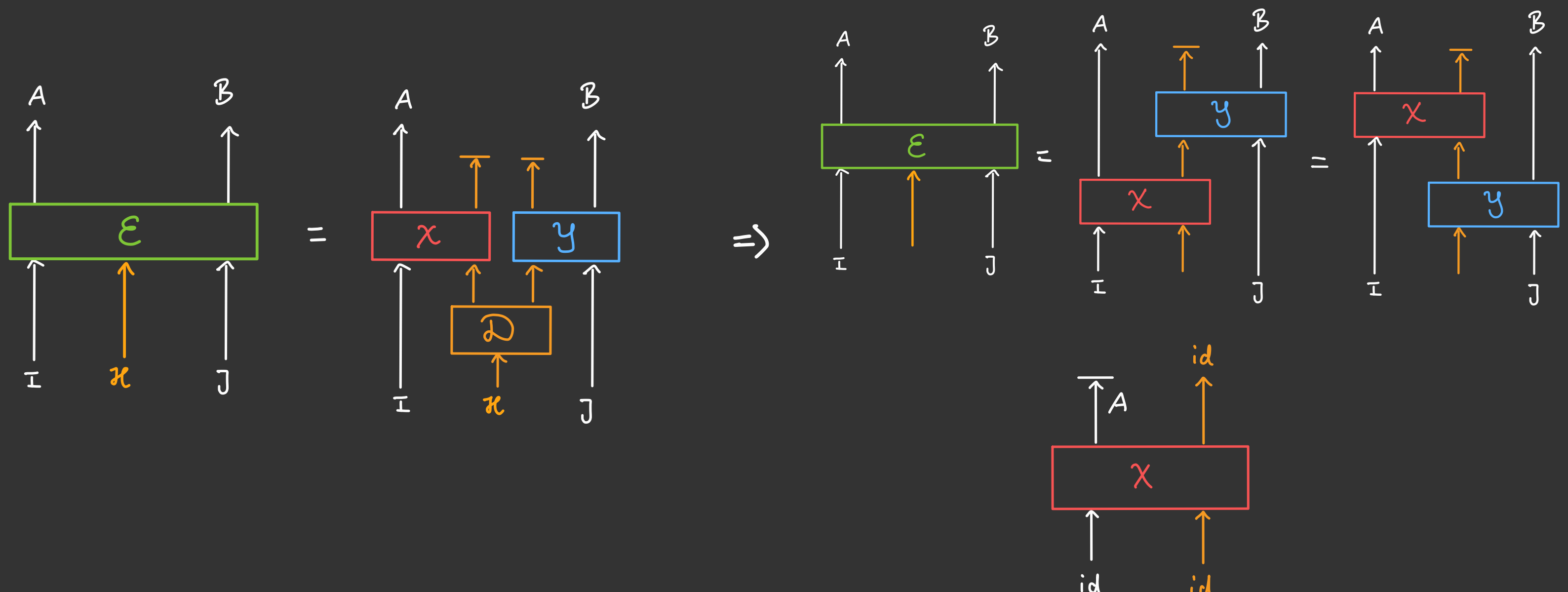


FACTORISATION

Main result (arXiv: 2308. 05792)

Converse theorem:

FACTORISATION $\Rightarrow$ COMMUTATION & UNITALITY

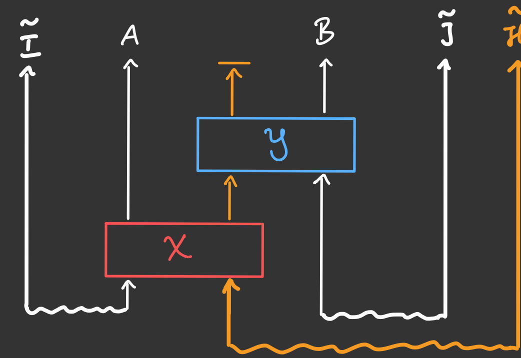


Main result (arXiv: 2308.05792)

Proof idea:

(1) Apply Choi-Jamiolkowski isomorphism:

$$\rho_{\tilde{I}\tilde{J}\tilde{\mathcal{H}}AB} :=$$



(2) Use commutation & unitality to show $I(A\tilde{I} : B\tilde{J} \mid \tilde{\mathcal{H}})_\rho = 0$.

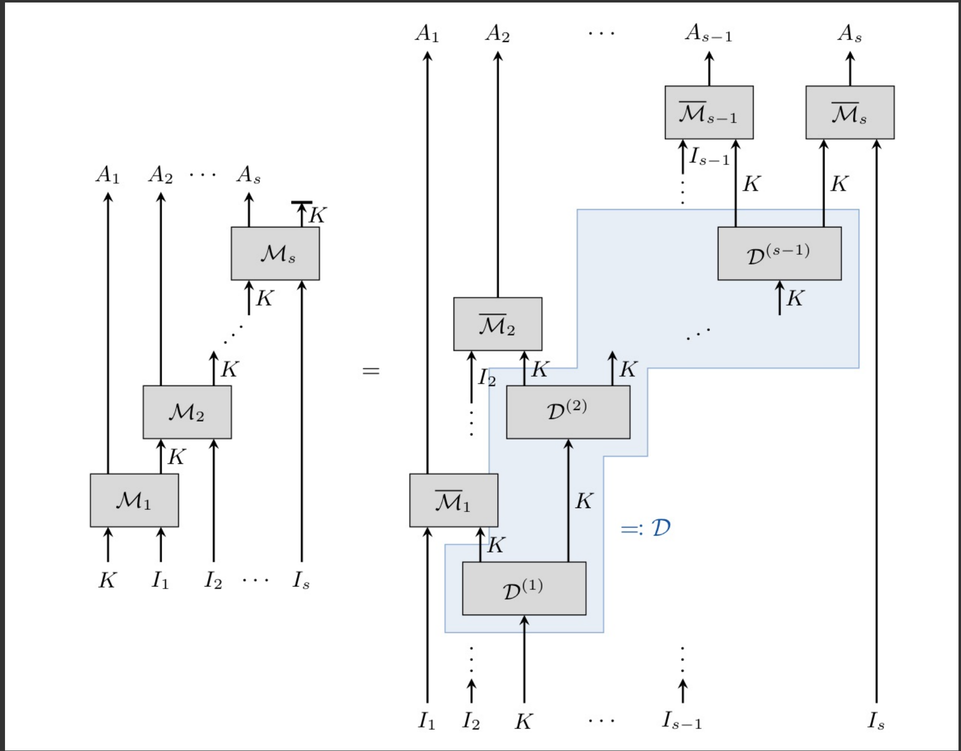
(3) Use result of [Haydn, Josza, Petz, Winter, 2004] to show

$$\rho_{\tilde{I}\tilde{J}\tilde{\mathcal{H}}AB} = \sum_z p_z \, \rho_{A\tilde{I}\tilde{\mathcal{H}}_A^z} \otimes \rho_{\tilde{\mathcal{H}}_B^z \tilde{J} B} .$$

(4) Apply C.-J. isomorphism backwards to obtain factorised maps.

Two observations

1) Generalisation to many maps:



2) Insight into Tsirelson's problem:

Instead of $[X_{i,a}, Y_{j,b}] = 0$ it is sufficient to have

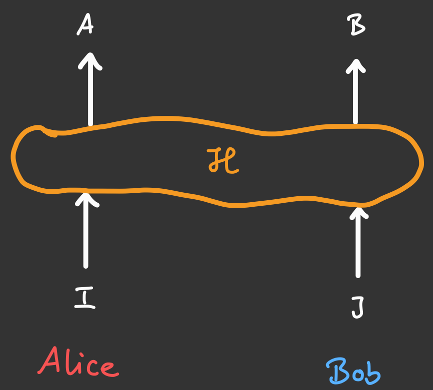
$$\sum_a \sqrt{X_{i,a}}\, Y_{j,b} \sqrt{X_{i,a}} = Y_{j,b} \qquad \forall\, i,j,b$$

What's next?

The problem with quantum gravity

$$\mathcal{H} \stackrel{?}{=} \mathcal{H}_A \otimes \mathcal{H}_B$$

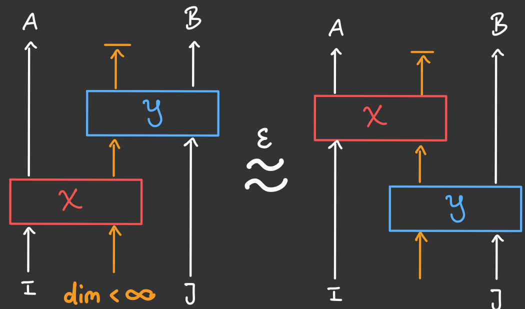
Can we construct commuting maps?



We do not expect the Hilbert space of quantum gravity to be naturally equipped with a tensor product structure such as (1.1). An essential property of the tensor product is that operators within the different factors commute: $[A \otimes I, I \otimes B] = 0$ for any operators A and B. There are strong arguments that such locally supported commuting operators simply cannot exist in quantum gravity [10, 11]. The basic obstruction is diffeomorphism invariance: physical operators in quantum gravity should be diffeomorphism invariant, but the action of a diffeomorphism is to permute the points of spacetime. Thus an invariant

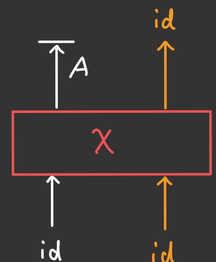
Donnelly, Freidel, 2016

Open question: Robust version?



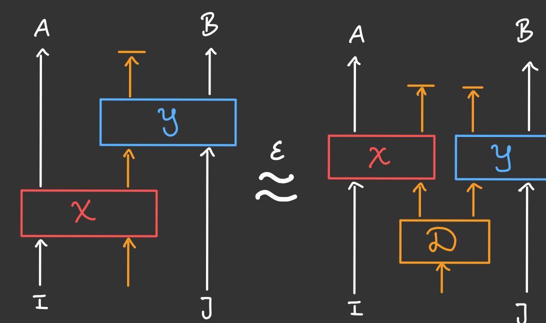
ε- COMMUTATION

and



UNITALITY

? ⟹



ε- FACTORISATION