Computing F-symbols for the center of a fusion category

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Joint work with
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Why F-symbols ?

Fusion category $\mathcal{C}$:

* Finite set of simple objects:

$$\mathrm{Ob}(\mathcal{C}) = \{ \mathbb{1}, a, b, c, \dots \}$$

* Morphisms $f: a \to b$, $g: b \to c$

  with composition

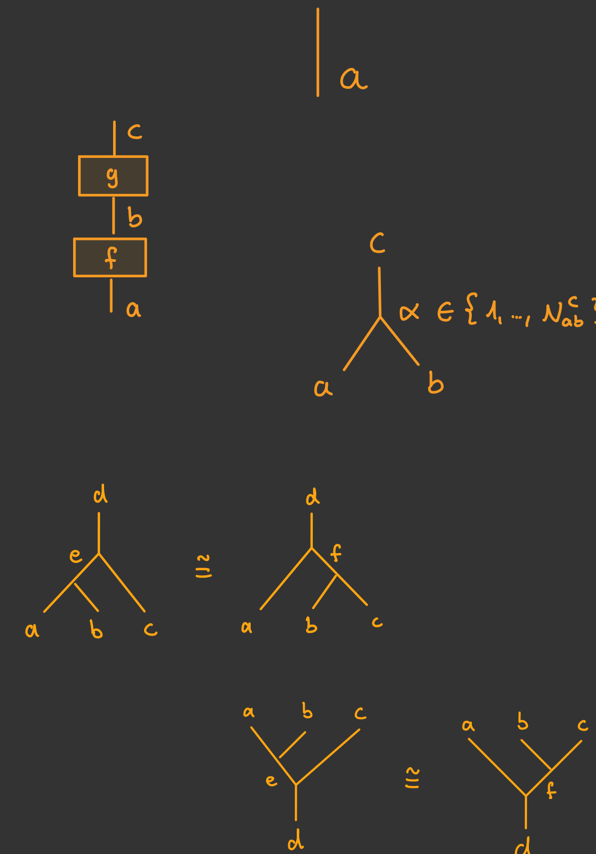
* Fusion rules: $a \otimes b = \sum_c N_{ab}^c \, c$

  ↳ Fusion spaces: V_{ab}^c with $\dim(V_{ab}^c) = N_{ab}^c$

* F-symbols (associator):

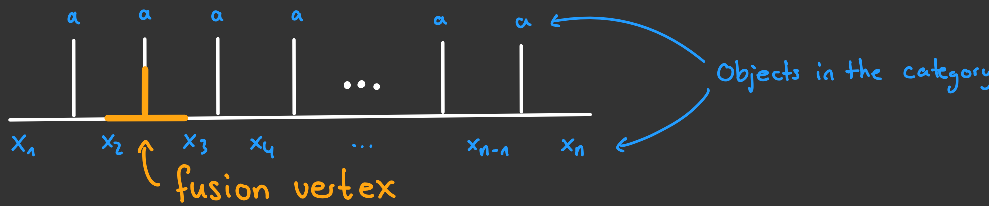
$$F^d_{abc} : \bigoplus_e V^e_{ab} \otimes V^{cd}_e \xrightarrow{\sim} \bigoplus_f V^d_{af} \otimes V^f_{bc}$$

Graphical calculus:



Application 1: Anyon chains

* One-dimensional lattice model:



* Dynamics: Nearest-neighbor interaction

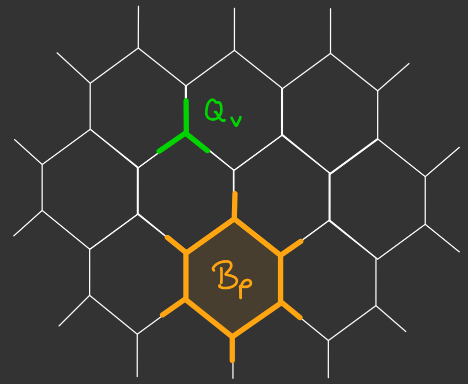
↳ Projection onto a simple object e : $P^{(e)} =$

$$P_i^{(e)} = \left(F^{x_{i-1}\, a\, a}_{x_{i+1}}\right)_{e\, x_i} \left(F^{x_{i-1}\, a\, a}_{x_{i+1}}\right)^{\dagger}_{x_i'\, e}$$

* Investigate phase transitions ⟶ conformal field theories ?

Application 2 : Levin-Wen model

* Two-dimensional lattice model :



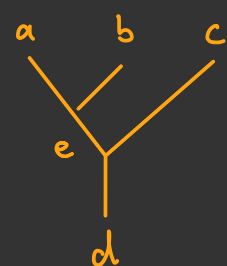
* Dynamics: Vertex operator Q_v
  Plaquette operator B_p contains lots of F-symbols

* Study of topological phases

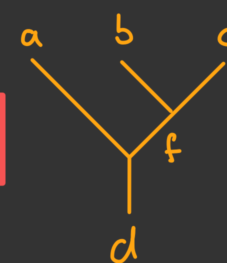
* Excitations of the model correspond to objects in the center of the category

Both applications show:

We <u>need</u> explicit data of the fusion categories, in particular F-symbols



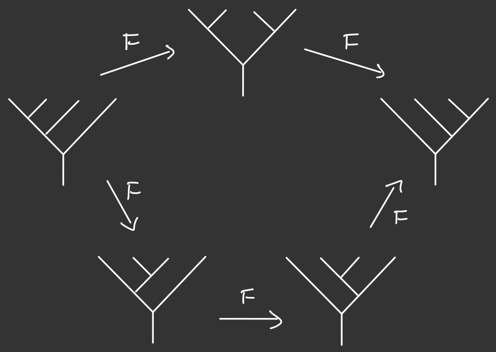
$$= \sum_f \boxed{\left(F^{abc}_d\right)_{ef}}$$



Challenges

Pentagon equation:

$$\left(F^{fcd}_{e}\right)_{gl}\left(F^{abl}_{e}\right)_{fk} = \sum_{h} \left(F^{abc}_{g}\right)_{fh}\left(F^{ahd}_{e}\right)_{gk}\left(F^{bcd}_{k}\right)_{hl}$$



with multiplicities:

$$\sum_{\delta} \left(F^{fcd}_{e}\right)_{(\beta,g,\gamma)(\delta,l,\nu)} \left(F^{abl}_{e}\right)_{(\alpha,f,\delta)(\lambda,k,\mu)}$$

$$= \sum_{h,\sigma,\psi,\rho} \left(F^{abc}_{g}\right)_{(\alpha,f,\beta)(\sigma,h,\psi)} \left(F^{ahd}_{e}\right)_{(\sigma,g,\gamma)(\lambda,k,\rho)} \left(F^{bcd}_{k}\right)_{(\psi,h,\rho)(\mu,l,\nu)}$$

* multivariate polynomial equations up to 3rd order
* thousands of variables + equations

The idea

based on joint work with Daniel Barter and Jacob Bridgeman,
SciPost Physics 13, 029 (2022)

Morita equivalence class:

* For a bimodule $\mathcal{C} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{D}$:

$$\mathrm{Rep}(\mathrm{Tub}_{\mathcal{C}}(\mathcal{M})) \equiv \mathcal{C}^*_{\mathcal{M}} \equiv \mathcal{D}$$

category of endomorphisms of $\mathcal{M}$ over $\mathcal{C}$

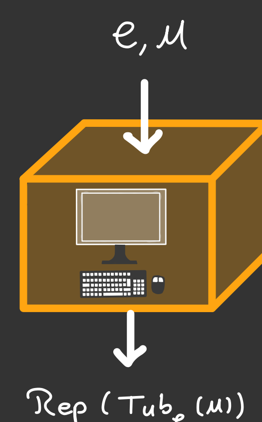
* If $\mathcal{C}$ and $\mathcal{M}$ are unitary $\Rightarrow$ $\mathcal{C}^*_{\mathcal{M}}$ unitary

GOAL: Construct the category $\mathrm{Rep}(\mathrm{Tub}_{\mathcal{C}}(\mathcal{M}))$ and calculate F-symbols

Why is this better?

$\mathrm{Rep}(\mathrm{Tub}_{\mathcal{C}}(\mathcal{M}))$ has more structure

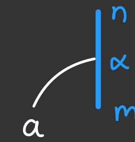
$\rightsquigarrow$ Compute F-symbols via solving <u>linear</u> equations



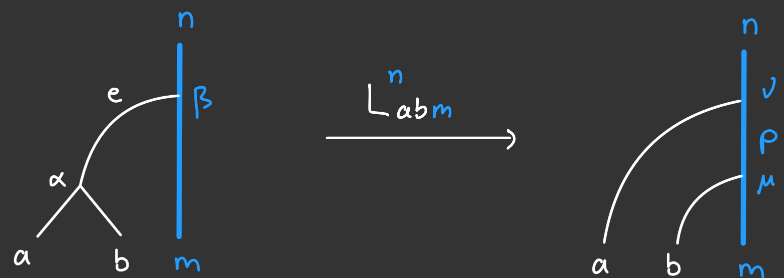
Tube algebra

Module category $\mathcal{C} \curvearrowright \mathcal{M}$

* Simple objects: $m, n \in \mathrm{Ob}(\mathcal{M})$, $a \in \mathrm{Ob}(\mathcal{C})$:



* Associator:



Pentagon equation:

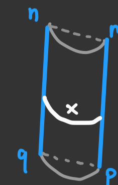
$$\left(L^{n}_{fcm}\right)_{gp}\left(L^{n}_{abp}\right)_{fq} = \sum_{z}\left(F^{g}_{abc}\right)_{fz}\left(L^{n}_{azm}\right)_{gq}\left(L^{q}_{bcm}\right)_{zp}$$

↳ only quadratic polynomials (if F-symbols of $\mathcal{C}$ are known)

Module tube category $\mathrm{Tub}_{\mathcal{C}}(\mathcal{M})$

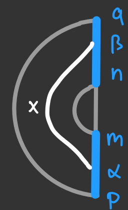
* Simple objects: $\quad Ob(\mathrm{Tub}_{\mathcal{C}}(\mathcal{M})) = \{(m,n) \mid m,n \in Ob(\mathcal{M})\}$

* Morphisms: $\quad \mathrm{Hom}_{\mathrm{Tub}_{\mathcal{C}}(\mathcal{M})}((m,n),(p,q))$ "half tubes":



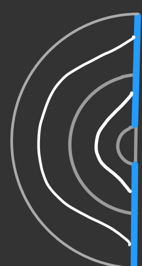
basis:

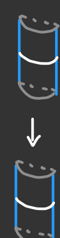
$$\Lambda = \Bigg\{$$



$$\Bigg| \quad x \in Ob(\mathcal{C}), \quad 1 \leq \alpha \leq N_{xm}^{p}, \quad 1 \leq \beta \leq N_{xn}^{q} \Bigg\}$$

Composition:

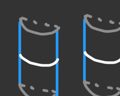




"stacking tubes"

tube algebra

tensor product:



The algorithm

Step 1: Construct irreducible representations of the module tube algebra

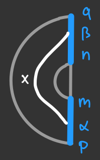
* $\mathcal{M}$ irreducible $\Rightarrow$ $\quad \mathrm{Tube}_{\mathcal{e}}(\mathcal{M}) \cong \bigoplus_{\alpha=1}^{n} \underbrace{\mathrm{Mat}(D_\alpha)}_{= D_\alpha \times D_\alpha \text{ - matrix algebra over } \mathbb{C}}$

* convenient to find matrix unit basis:

$$\left\{ [e_\alpha]_{ij} \;\middle|\; 0 \le i,j < D_\alpha \,,\; [e_\alpha]_{ij}\,[e_\alpha]_{kl} = \delta_{jk}\,[e_\alpha]_{il} \right\}$$

* and express them in terms of tube diagrams:

$$[e_\alpha]_{ij} = \sum_{P \in \Lambda} c_P^\alpha \, P$$



* irreducible representation: vector space + action

  Natural choice: Algebra acts on itself

  ↳ irrep: vector space V^α with basis $[v_\alpha]_i = [e_\alpha]_{i0}$

  $=$ 

* irreps $V^\alpha \longrightarrow$ simple objects in $\mathcal{e}^*_{\mathcal{M}}$

Step 2: Compute fusion rules

* Construct basis for tensor product space

$\alpha, \beta \in \mathrm{Ob}(\mathcal{C}^*_\mu)$: $\alpha \otimes \beta =$  $\longrightarrow$ $[v_\alpha]_i \otimes [v_\beta]_j = \sum \cdots$ 

objects have to match here

$\dim(\alpha \otimes \beta) \leq \dim \alpha \cdot \dim \beta$

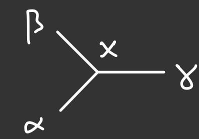
* decompose into irreps:  $\cong \bigoplus_\gamma$ 

* Compute fusion rules: Take generic vector in $\alpha \otimes \beta$: $v = \sum_{i,j} c_{ij}$ 

Project onto γ using $\mathbb{1}_\gamma = \sum_i [e_\gamma]_{ii}$

$N^\gamma_{\alpha\beta}$ = dimension spanned by such projected vectors

Step 3: Compute embedding matices

* Map that embeds γ into $\alpha \otimes \beta$: $\quad V^{\gamma;x}_{\alpha\beta} :=$ 

$= \dim(\alpha \otimes \beta) \times \dim \gamma$ matrix

* choose $w \in \alpha \otimes \beta$

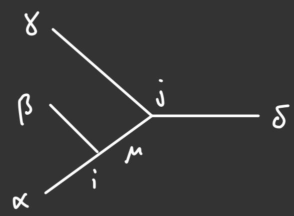
$\downarrow$

$$[e_\gamma]_{00}\, w =: v^{\gamma}_{0,x} \qquad v^{\gamma}_{1,x} = [e_\gamma]_{10}\, v^{\gamma}_{0,x}$$

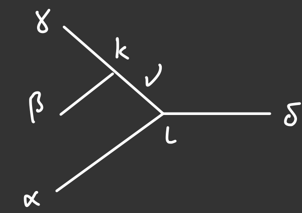
$$\begin{pmatrix} v^{\gamma}_{0,x} & v^{\gamma}_{1,x} & \cdots \end{pmatrix} = V^{\gamma;x}_{\alpha\beta}$$

* Reshape into 3-tensor of size $(\dim \alpha, \dim \beta; \dim \gamma)$

Step 4: Compute F-symbols



$$= \sum_{k,\nu,l} \left(F^{\delta}_{\alpha\beta\gamma}\right)_{(i,\mu,j)(k,\nu,l)}$$



$$\| \qquad\qquad\qquad\qquad \|$$

$$V^{\mu;i}_{\alpha\beta} \; V^{\delta;j}_{\mu\gamma} \qquad\qquad V^{\delta;l}_{\alpha\nu} \; V^{\nu;k}_{\beta\gamma}$$

$$\boxed{\text{Linear equation}}$$

Compare pentagon equation:

$$\sum_{\delta} \left(F^{fcd}_{e}\right)_{(\beta,g,\gamma)(\delta,l,\nu)} \left(F^{abl}_{e}\right)_{(\alpha,f,\delta)(\lambda,k,\mu)} = \sum_{h,\sigma,\psi,\varsigma} \left(F^{abc}_{g}\right)_{(\alpha,f,\beta)(\sigma,h,\psi)} \left(F^{ahd}_{e}\right)_{(\sigma,g,\gamma)(\lambda,k,\varsigma)} \left(F^{bcd}_{k}\right)_{(\psi,h,\varsigma)(\mu,l,\nu)}$$

Summary:

1. Find irreducible representations of the tube algebra $\mathrm{Tub}_e(\mathcal{M})$
   ↳ Simple objects

2. Compute the decomposition of the tensor product of all irrep pairs
   ↳ Fusion rules

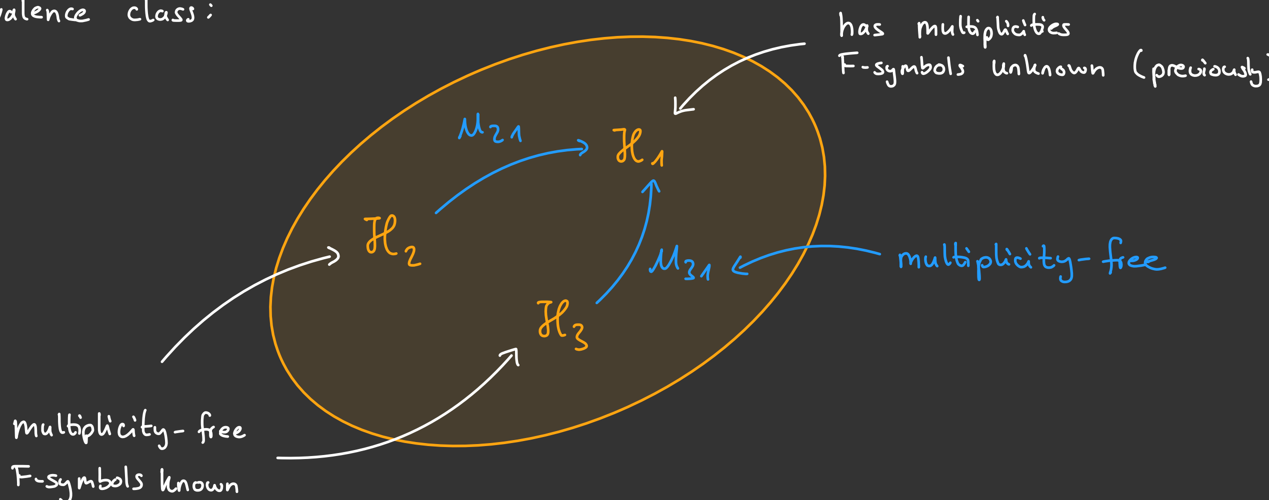
3. Form explicit matrices for embeddings, reshape into 3-tensors

4. Solve linear equations to compute F-symbols

Applications

Application: Haagerup Categories

Morita equivalence class:



$\hookrightarrow$ construct $\mathrm{Rep}\left(\mathrm{Tub}_{\mathcal{H}_3}(\mathcal{M}_{31})\right) \equiv \mathcal{H}_1$

Example

<u>Example:</u> $\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^*_{\mathrm{Vec}} \equiv \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

* $\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})$:
  - $\mathrm{Obj} = \{0, 1\}$
  - $d_0 = d_1 = 1$
  - $a \otimes b = a+b \mod 2$
  - all F-symbols $= 1$ when allowed

* Module category Vec:
  - $\mathrm{Obj} = \{*\}$
  - $d_* = \sqrt{2}$
  - all L-symbols $= 1$ when allowed

* $\mathrm{Tub}_{\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})}(\mathrm{Vec})$:

$\Lambda = \{ T_0 =$  , $T_1 =$  $\}$

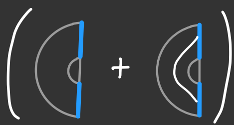
$$T_x \circ T_y = T_{x+y \mod 2}$$

Example: $\quad \mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^*_{\mathrm{Vec}} \cong \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

Step 1: Find irreducible representations of $\mathrm{Tub}_{\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})}(\mathrm{Vec})$

$$\mathrm{Tub}_{\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})}(\mathrm{Vec}) \cong \mathbb{C} \oplus \mathbb{C}$$

1-dim. matrix algebras

Matrix units: $\quad [e_1]_{00} = \frac{1}{2}(T_0 + T_1) = \frac{1}{2}\Big($  $\Big)$

$[e_\psi]_{00} = \frac{1}{2}(T_0 - T_1) = \frac{1}{2}\Big($  $\Big)$

Basis for irreps: $\quad [v_1] = [e_1]_{00} = 1$ 

$[v_\psi] = [e_\psi]_{00} = \psi$ 

action: $[e_\alpha][v_\beta] = \delta_{\alpha\beta}[e_\alpha]$

Example: $\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^*_{\mathrm{Vec}} \equiv \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

Step 2: Compute Fusion rules

Tensor product basis:

$[V_1] \otimes [V_1] = \frac{1}{4}\Big($  $+$  $+$  $+$  $\Big)$

$[V_1] \otimes [V_\psi] = \frac{1}{4}\Big($  $-$  $+$  $-$  $\Big)$

$[V_\psi] \otimes [V_1] = \frac{1}{4}\Big($  $+$  $-$  $-$  $\Big)$

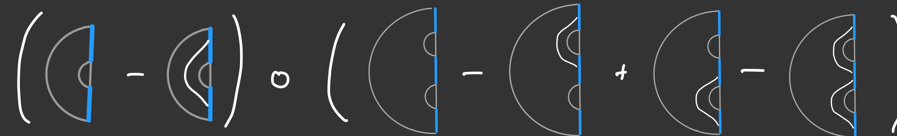
$[V_\psi] \otimes [V_\psi] = \frac{1}{4}\Big($  $-$  $-$  $+$  $\Big)$

Example: $\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^*_{\mathrm{Vec}} \simeq \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

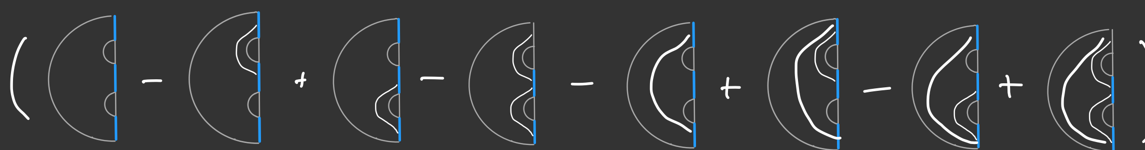
Step 2: Compute Fusion rules

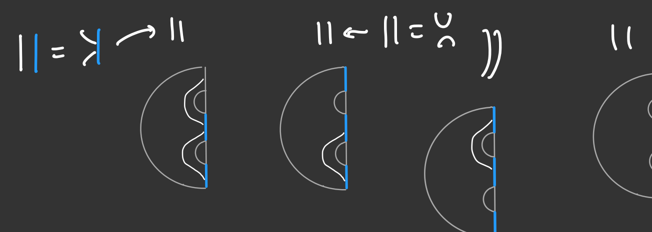
Example: Project ψ into $1 \otimes \psi$:

$[e_\psi]_{00}\left([V_1] \otimes [V_\psi]\right) = \frac{1}{8}$



$= \frac{1}{8}$





$= [V_1] \otimes [V_\psi] \Rightarrow 1 \otimes \psi = \psi$

$\psi \otimes 1 = \psi$

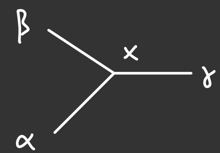
$\psi \otimes \psi = 1$

$\boxed{\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})}$

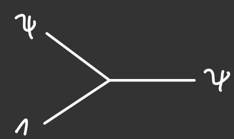
Example: $\mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^{*}_{\mathrm{Vec}} \simeq \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

Step 3: Compute embedding matrices

$V^{\gamma;x}_{\alpha\beta}$:=



Example: $V^{\psi}_{1\psi}$ =



(1-dim.)

choose a general vector $w \in 1 \otimes \psi$: $w = c \cdot [v_1] \otimes [v_\psi]$

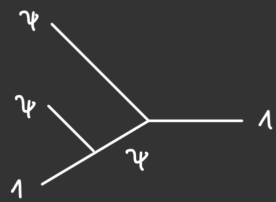
$$[e_\psi]\left(c \cdot [v_1] \otimes [v_\psi] \right) = c \cdot [v_1] \otimes v_\psi]$$

$$\Rightarrow V^{\psi}_{1\psi} = \left(w^{\psi}_{1\psi} \right) \cdot 2^{1/4} \qquad \text{(isometry)}$$

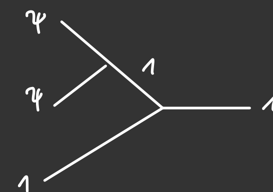
$|w^{\psi}_{1\psi}| = 1$

Example: $\quad \mathrm{Vec}(\mathbb{Z}/2\mathbb{Z})^*_{\mathrm{Vec}} \equiv \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$

Step 4: Compute F-symbols



$$= \left(F^{1\psi\psi}_{1}\right)_{\psi 1}$$



$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

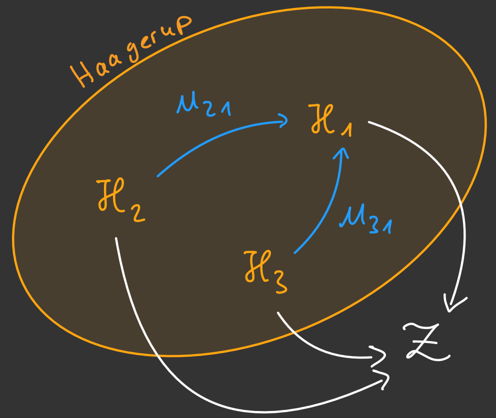
$$V^{\psi}_{1\psi}\; V^{1}_{\psi\psi} \qquad\qquad\qquad\qquad V^{11}_{1}\; V^{\psi\psi}_{1}$$

$$\Rightarrow \left(F^{1\psi\psi}_{1}\right)_{\psi 1} = \frac{V^{\psi}_{1\psi}\, V^{1}_{\psi\psi}}{V^{11}_{1}\, V^{\psi\psi}_{1}} = \frac{\omega^{\psi}_{1\psi}\, \omega^{1}_{\psi\psi}}{\omega^{1}_{11}\, \omega^{\psi\psi}_{1}} = \frac{\omega^{1\psi}_{\psi}}{\omega^{11}_{1}}$$

We can set all parameters $\omega^{\gamma}_{\alpha\beta} = 1$.

↳ All allowed F-symbols are equal to 1

$\boxed{\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})}$



The center

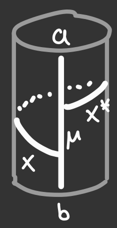
Ongoing work with Jacob Bridgeman and Alexander Hahn

Similar idea: $\mathrm{Rep}(\mathrm{Tub}(\mathcal{C})) = \mathcal{Z}(\mathcal{C})$

different kind of tube

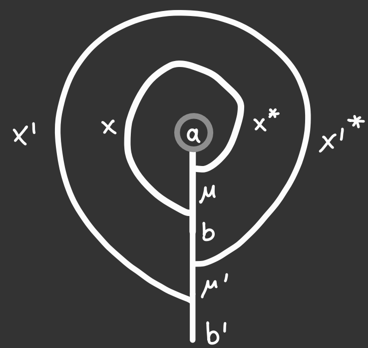
center of $\mathcal{C}$

Full tubes:



$\rightsquigarrow$ $T[abx\mu] :=$

Composition: Put one tube around the other



$= \sum \ldots$ (F-symbols etc.)

$\Rightarrow$ We have an algebra, so we can do step 1 from the algorithm:

Calculate its irreducible representations to get the simple objects of the center

Step 2: Calculate fusion rules

→ we need a basis for the tensor product space

Tensor product of two tubes:

Why would we need this?

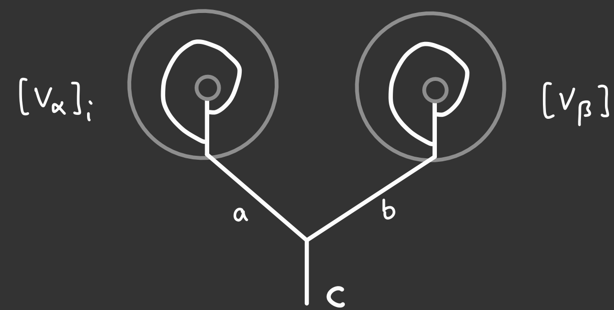
Option 1

Tensor product of two irreps:
basis of the space is formed by

$$\{ [V_\alpha]_i \otimes [V_\beta]_j \;:\; [V_\alpha]_i \text{ - basis of irrep } \alpha,\; [V_\beta]_j \text{ - basis of irrep } \beta \}$$

$$\Rightarrow \dim(\alpha \otimes \beta) = \dim\alpha \cdot \dim\beta$$

Option 2



Tensor product of two irreps:
basis of the space is formed by

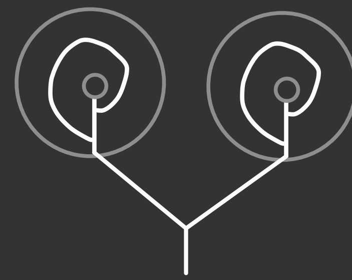
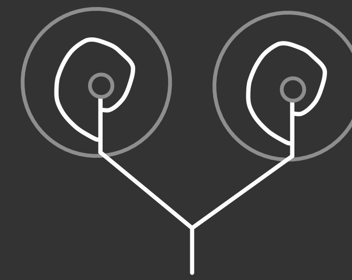
$$\{ [V_\alpha]_i \otimes_c [V_\beta]_j \;:\; [V_\alpha]_i \text{ - basis of irrep } \alpha,\; [V_\beta]_j \text{ - basis of irrep } \beta,\; c \in a \otimes b \}$$

$$\Rightarrow \dim(\alpha \otimes \beta) \geq \dim\alpha \cdot \dim\beta$$

How to get fusion rules, e.g. $\alpha \otimes \beta = \gamma$

Recall: $[e_\gamma]_{00} \left([V_\alpha]_i \otimes [V_\beta]_j \right)$

$\Sigma \ldots$ 

$\Sigma \ldots$  $= \Sigma \ldots$ 

Can compose these!

embedding of γ into $\alpha \otimes \beta$

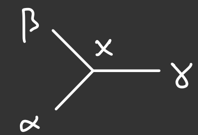
What we got: Correct fusion rules for the center of

* Fibonacci
* Vec $\mathbb{Z}2$
* Vec $S3$

Haagerup? Can't solve it on a laptop ...

Step 3: Compute embedding matrices

* Map that embeds γ into $\alpha \otimes \beta$: $\quad V^{\gamma;x}_{\alpha\beta} :=$

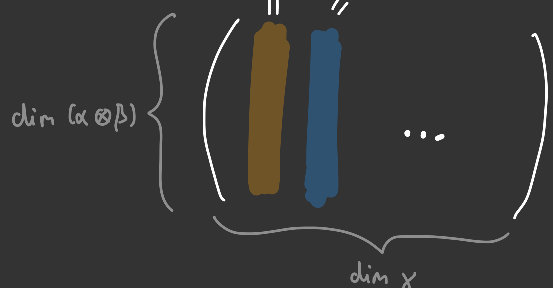


$= \dim(\alpha\otimes\beta) \times \dim\gamma$ matrix

* choose $w \in \alpha \otimes \beta$

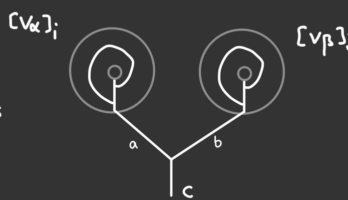
$\downarrow$

$[e_\gamma]_{00}\, w =: v_0^\gamma \qquad v_1^\gamma = [e_\gamma]_{10}\, v_0^\gamma$



$= V^\gamma_{\alpha\beta}$

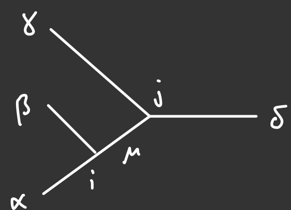
$v_0^\gamma = \sum_i c_i \, [v_{\alpha\otimes\beta}]_i \longrightarrow$ only works if we treat each of those vectors as an individual summand



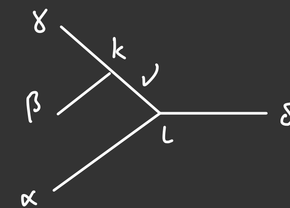
* Reshape into 3-tensor of size $(\dim\alpha, \dim\beta; \dim\gamma) \longleftarrow$ does not work if $\dim(\alpha\otimes\beta) > \dim\alpha \cdot \dim\beta$

Step 4: Compute F-symbols

How?



$$= \sum_{k,\nu,l} \left(F^{\delta}_{\alpha\beta\gamma}\right)_{(i,\mu,j)(k,\nu,l)}$$



$$\parallel \qquad\qquad \parallel$$

$$V^{\mu;i}_{\alpha\beta}\, V^{\delta;j}_{\mu\gamma} \qquad\qquad V^{\delta;l}_{\alpha\nu}\, V^{\nu;k}_{\beta\gamma}$$

Additional challenge: Computationally costly to compute embedding matrices