AIM, 26.09.22

A physicist's view on fusion categories

1) What is a fusion category? ⟶ A collection of diagrams we can do calculations with

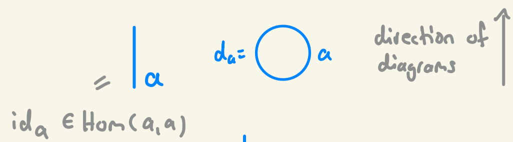
2) Why do we care? ⟶ We can learn about properties of physical theories

1. What is a fusion category?

Def: A fusion category $\mathcal{C}$ over k is a k-linear ($\mathbb{C}$-) rigid semisimple monoidal category with finitely many simple objects and finite-dimensional morphism spaces such that the identity object is simple.

⟶ translate into diagrams & data

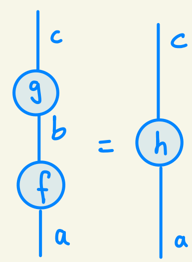
Graphical representation



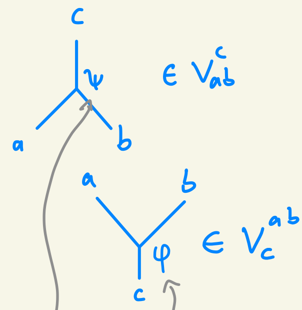
* Finite set of simple objects
  $Obj(\mathcal{C}) = \{1, a, b, \dots\}$ ← identity object
  with $d_a = \dim(a)$

* Morphisms: $f \in \underbrace{Hom(a,b)}_{\substack{\text{finite-dimensional} \\ \mathbb{C}\text{-vector space}}}$

  \+ composition. $g \in Hom(b,c)$
  $\Rightarrow h = g \circ f \in Hom(a,c)$



* Fusion ring: $a \otimes b = \bigoplus_{c \in Obj(\mathcal{C})} N_{ab}^{c}\, c$
  $N_{ab}^c = \dim(V_{ab}^c)$ — fusion multiplicities $\in \mathbb{N}$



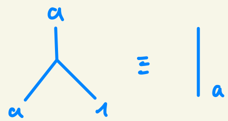
Vector spaces $V_{ab}^c = Hom(a \otimes b, c)$ "fusion"

Dual space: $V_c^{ab} = Hom(c, a \otimes b)$ "splitting"

don't need labels

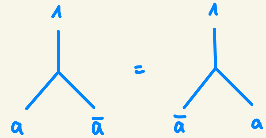
Multiplicities: If $N_{ab}^c \in \{0,1\}$, the category is called "multiplicity-free" ← use this from now on!

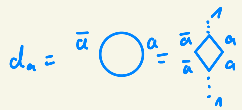
Special case: Fusion with unit object is always trivial → don't draw lines



* **Duals:** For every a, there is an $\bar{a}$:

$$a \otimes \bar{a} = \bar{a} \otimes a = 1$$





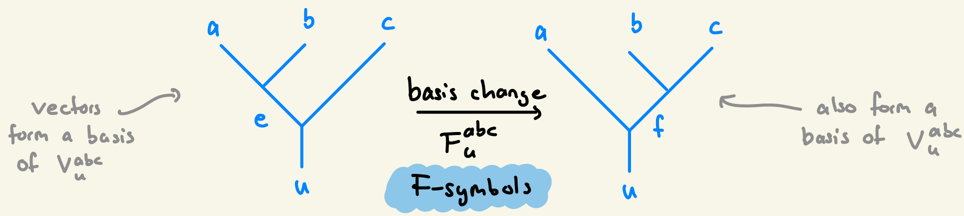
* **Semisimple:** Every object can be written as a sum of simple objects

→ only need to worry about simple objects

We're not done yet...

Build bigger vector spaces:

$$V_u^{abc} \cong \bigoplus_e V_e^{ab} \otimes V_u^{ec} \cong \bigoplus_f V_b^{af} \otimes V_f^{bc}$$

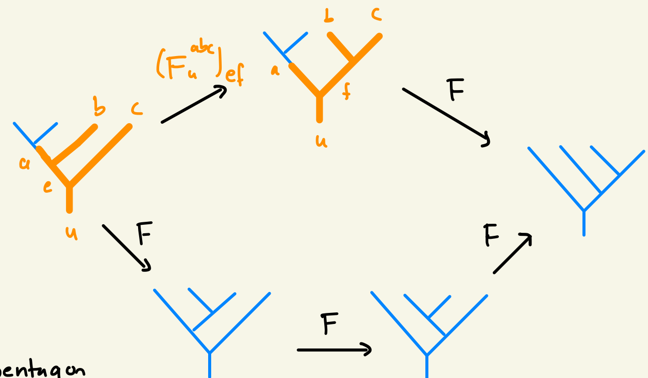


How to calculate the F-symbols?

↳ Solve the pentagon equation

$$(F_u^{abc})_{ef} \; (F_{\cdot}^{\cdot\cdot})_{\cdot\cdot} = \sum F F F$$

* $(\#\text{ simples})^6$ variables
* $(\#\text{ simples})^9$ equations



If a (or several) solutions to the pentagon equation exists, then the fusion category exists.

If the F-symbols are unitary (as matrices), the fusion category is unitary.

In summary: Fusion category data: – Simple objects
– Fusion rules
– F-symbols

Example: Fibonacci category Fib

* Obj (Fib) = $\{1, \tau\}$

* Fusion rules :

	1	τ
1	1	τ
τ	τ	$1+\tau$

$\longrightarrow$ multiplicity-free

* F-symbols: Easy calculation
Two solutions: unitary one (Fib)
non-unitary one (Yang-Lee)
} two different categories

2. Why do we care?

A) Physics: Describe the world ($\hat{=}$ the laws of nature) using theories
Particularly successful: (Quantum) field theories

But: Often difficult to construct field theory directly
↳ Instead, construct discrete lattice and study continuous limit

Fusion category $\xrightarrow{\text{construct}}$ Discrete theory on a lattice $\xrightarrow{\text{limit}}$ Continuous (quantum) field theory

1D chain ("Golden chain")	Conformal field theory
2D Lattice ("Levin-Wen model")	Topological quantum field theory

B) Topological quantum computin with anyons

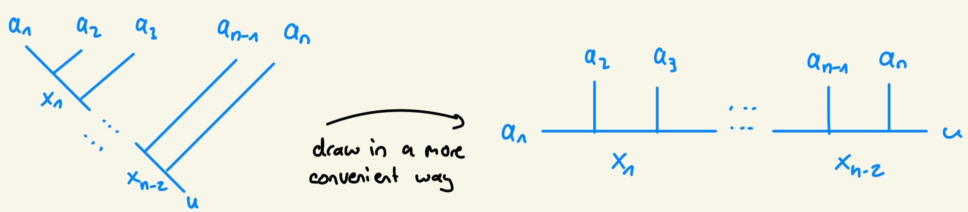
$$\text{(UFCs)}\quad \text{Unitary fusion categories} \xrightarrow[\text{Quantum double}]{\text{Drinfeld center}} \text{Unitary modular tensor categories} \quad \text{(UMTCs)}$$

Now: How do we use fusion categories in these applications and what are typical problems?

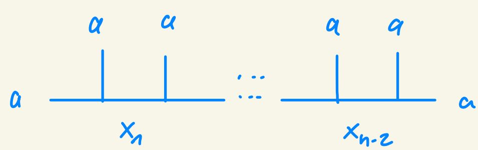
A) 1D chain from a unitary fusion category

We need: 1) Hilbert space (set of states)
2) Hamiltonian (dynamics)

Consider vector space $V_u^{a_1 a_2 \dots a_n}$:



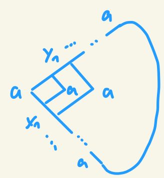
Simplify the chain:



$\in V_a^{a..a}$

* Basis vectors: $|x_1 \dots x_{n-2}\rangle$
given by fusion rules: $x_1 = a \otimes a$
$x_2 = x_1 \otimes a$
$\vdots$
$x_{n-2} = x_{n-3} \otimes a$

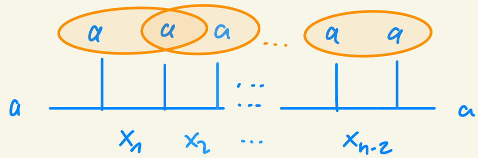
* Inner product given by trace: $\langle y_1 \dots y_{n-2} | x_1 \dots x_{n-2}\rangle =$

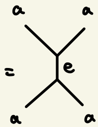


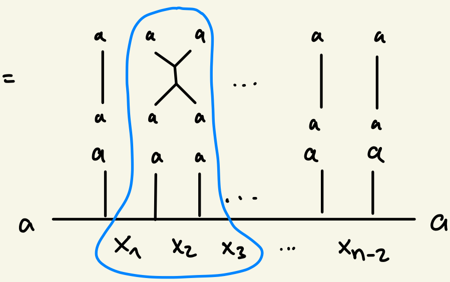
(we usually work in a spherical category; unitary => spherical?)

$\boxed{\text{Hilbert space}}$ ✓

Dynamics: Nearest-neighbor interaction : Fusion



Project onto a specific fusion channel: $P^{(e)} =$  $\cdot \frac{1}{n} = \frac{\sqrt{d_e}}{d_a}$ (normalization)

$P_2^{(e)} =$ 

Simplify this using F-symbols (bring back into "standard form")

$$\Rightarrow P_2^{(e)} = \left(F^{x_1 a a}_{x_3}\right)_{x_2 e} \left(F^{x_1 a a}_{x_3}\right)^{\dagger}_{\tilde{x}_2 e}$$

need explicit matrices for F-symbols

Local Hamiltonian: $H_i = \sum_e \gamma_e P_i^{(e)}$

Global Hamiltonian: $H = \sum_i H_i$

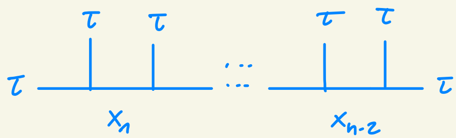
same Hamiltonian for all i

$\boxed{\text{Dynamics}}$ ✓

Now: Do numerics to study limit $n \to \infty$ of the chain and get information about the corresponding physical theory:

- Does the system exhibit a phase transition? → sign of CFT
- What central charge does the CFT have?

Example: Fibonacci chain ("Golden chain")



Basis of Hilbert space:

$x_1 = \tau \otimes \tau = 1 + \tau$ (can be 1 or τ)

$x_2 = x_1 \otimes \tau$ If $x_1 = 1 \Rightarrow x_2 = \tau$

$\Rightarrow$ $|1\tau \ldots\rangle$

$|\tau 1 \ldots\rangle$

$|\tau\tau \ldots\rangle$

~~$|11 \ldots\rangle$~~ ← not this one

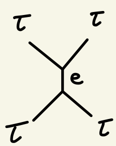
Dimension of the chain:

#τs on the outside

$n+1$	4	5	6	7	8	...
dim	2	3	5	8	13	...

dimension grows quickly (recall: $n \to \infty$)

Projectors:



Fusion rule: $\tau \otimes \tau = 1 + \tau$

↳ $e = 1, \tau$

Hamiltonian: $H_i = \gamma_1 P_i^{(1)} + \gamma_\tau P_i^{(\tau)}$ $\gamma_1, \gamma_\tau \in \mathbb{R}$

infinitely many possibilities; which to choose?

All of these are not too bad for this small category; $n \to 240$

$H_i = -P_i^{(1)} \rightarrow$ Phase transition ✓

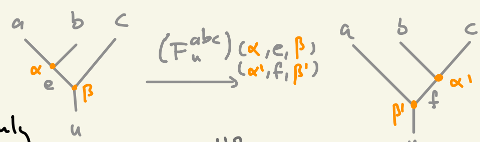
$c = 7/10$ (tricritical Ising model)

$H_i = -P_i^{(\tau)} \rightarrow$ Phase transition ✓

$c = 4/5$ (critical 3-state potts model)

found known CFTs ✓

Biggest problem:



* Hilbert space dimension grows too quickly

- big category (more simple objects)
- multiplicities (larger label space) $e \rightarrow \{\alpha, e, \beta\}$

H3: # simples $= 6$ $n \to 30$ [$n \to 144$]

simples $= 3$ + mult. $n \to 100$ (but no conclusive results)

$\frac{1}{2} E_6$

* can't even solve pentagon equations

B) Topological Quantum Computing

Idea:

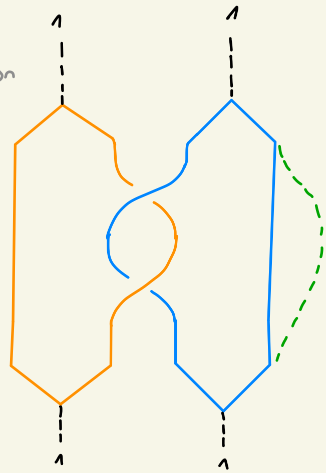
3. Fuse anyons to vacuum

fusion outcome gives result of computation

2. Implement gates via braiding

new concept → UMTC

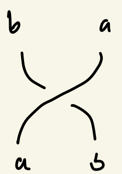
1. Create anyons



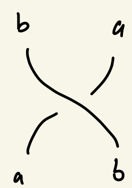
robust against noise;

Braiding:

$b_{a,b}$:



$b_{a,b}^{-1}$:



→ Fulfil hexagon equation (consistency of braiding & F-symbols)

→ This yields a universal topological quantum computer

For example: Fibonacci anyons can approximate a universal set of gates with arbitrary precision

Problem: Even simple gates can be complicated to implement; require thousands of braiding operations

Fibonacci anyons have not yet been created in the lab

Find better candidates?

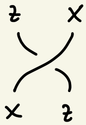
How to get a UMTC?

$$\text{UFC} \xrightarrow[\text{Drinfeld center}]{} \text{UMTC}$$

?

Drinfeld center $\mathcal{Z}(\mathcal{C})$ ← UFC

Objects: (z, γ), where $z \in \mathrm{Obj}(\mathcal{C})$ (not simple!)

and $\gamma_X =$ 

$\forall\, X \in \mathcal{C}$ "half-braiding"

that fulfill a consistency condition similar to ~~pentagon~~ hexagon equation

⇒ Gives a unitary modular tensor category

How to construct it?

1) Solve equations arising from definition → Quickly gets very difficult
2) Levin-Wen model: Same (lots of complicated equations)
3) Representations of tube algebra: Maybe ok?

Takeaway: As a physicist working with fusion categories you spend a lot of time trying to solve a huge set of polynomial equations in a large number of variables.